Baby's First Bessel

Modeling time-dependent wave perturbations on a circular membrane, or,

Why trampoline bouncy

By Nikko Ong (April 3, 2026)

We will discuss the solution to the vibrating circular membrane problem, the derivation of the wave equation from Newton's Second Law of Motion, the derivation of the polar Laplacian, the separation of variables into 3 ordinary differential equations (ODEs), the Bessel equation, the derivation of the Bessel functions, and the solution coefficients of the wave equation.

Baby Version: These blue sections will appear after each section to help babies (and adults) understand the material in a more intuitive way. Easy word make big math look not scary.

1 The Wave Equation

The wave equation is a second-order linear partial differential equation that describes how disturbances propagate through a medium in time and space. It states that the acceleration of any point is proportional to that point's relative displacement compared to its neighbors.

We will show the derivation of the wave equation in one dimension from Newton's Second Law in Appendix Section A. We will see that the restoring force due to tension on a string is proportional to its curvature at that point. We will then extend the wave equation to higher dimensional surfaces, like our two dimensional membrane.

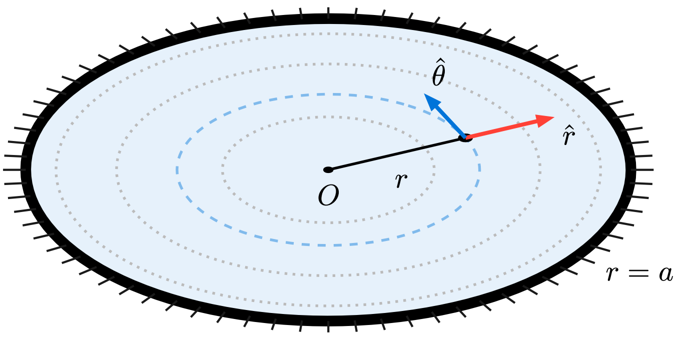
$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \text{The Wave Equation} \tag{1}$$

The constant c represents the propagation speed of a given disturbance in the medium and is defined as $\sqrt{\frac{T}{\rho}}$ where T is tension force and ρ is the surface density.

Baby Version: This math rule tells us how waves travel. It says that the way a wave changes shape as time passes is related to how it stretches and bends. It applies to ocean waves, sound waves, and to the way a trampoline moves when we bounce on it!

2 Boundary and Initial Conditions

We want to solve the 2D wave equation for the vertical displacement $u(r, \theta, t)$ of a circular membrane of radius a with a clamped boundary.



We observe a clamped edge at $r = a$ of the cylindrical membrane (we call this a Dirichlet condition), such that the outside rim is fixed in place and does not move:

$$u(a, \theta, t) = 0 \tag{2}$$

Our initial conditions (we call these Cauchy conditions) at time $t = 0$ are:

$$\begin{aligned} u(r, \theta, 0) &= f(r, \theta) \\ \frac{\partial u}{\partial t}(r, \theta, 0) &= g(r, \theta) \end{aligned} \tag{3}$$

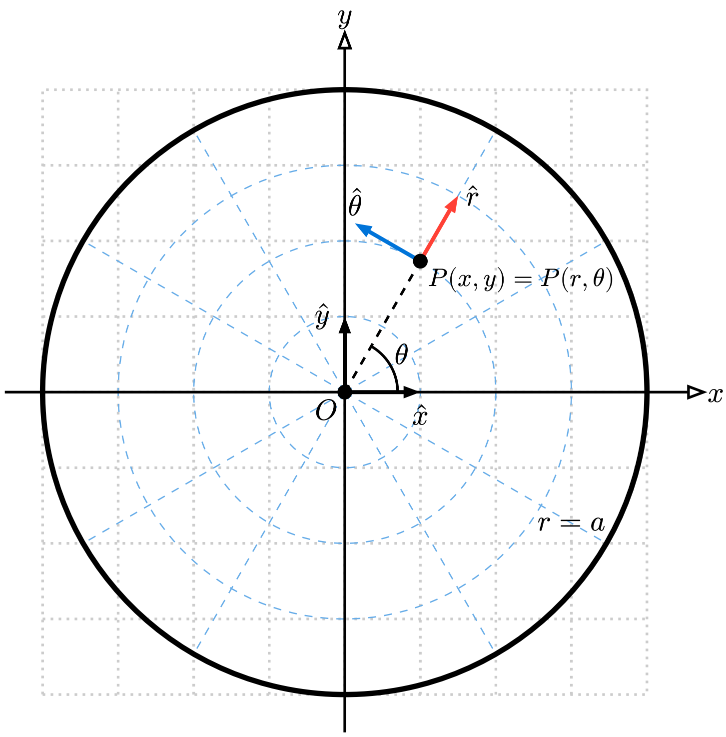
The initial conditions represents some initial displacement and surface velocity of the membrane, potentially from a bounce or a strike.

Baby Version: We are looking at a trampoline. The outside edge is held tight and cannot move at all. To figure out how the trampoline bounces and changes as time passes, we also need to know exactly what the trampoline looks like at the beginning and how fast it is moving right when we start watching it.

3 The Polar Laplacian

The Laplacian in Cartesian (x, y) coordinates ∇_c is given by:

$$\nabla_c^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{4}$$



In the appendix Section B, we will show that the polar Laplacian ∇_p in terms of (r, θ) is given by:

$$\nabla_p^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{5}$$

Baby Version: Instead of measuring the trampoline using a flat grid of squares, it is much easier to measure it using circles. We use a math trick to change our grid from squares into circles around the trampoline and lines from the center to the edge of the trampoline, like arms on a clock.

4 Separation of Variables

Using the polar Laplacian with Equation 1, we find wave equation in polar coordinates to be:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{6}$$

We use separation of variables to isolate variables on opposite sides of the equation so that we can attempt to solve this problem through direct integration. In doing so, we will decompose the linear PDE into coupled ODEs that we can solve analytically. With separation of variables, we can assume the solution to be of the form:

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t) \tag{7}$$

Substitute this solution template into the wave equation using the polar Laplacian and rearrange to get:

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2}\frac{\Theta''}{\Theta} \tag{8}$$

Because the left hand side of the equation is only expressed in terms of time t and the right hand side is only expressed in terms of space (r, θ), the equation only holds if both the left and right hand side are equal to some constant. We will conveniently choose $-\lambda^2$ as our constant.

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2 \tag{9}$$

We have cracked open the oyster. Now we just need to carefully separate, then recombine and enjoy.

The multivariate partial differential equation has almost been fully split into ordinary differential equations (ODEs), each given in terms of one variable that we can solve with the variety of tools in our toolbelt.

Baby Version: This big math problem of a bouncing trampoline is too hard to solve all at once. We use a trick to break the big problem into three smaller and easier pieces: one piece for time, one piece for the distance from the center, and one piece for the circles around the trampoline.

5 Time ODE

Looking at the leftmost term, the time term, and the rightmost term in the equation, we get the time ODE.

$$T'' + c^2\lambda^2 T = 0 \tag{10}$$

Solving this 2nd order differential equation (see Appendix Section C for the solution steps) gives us the following solution:

$$T(t) = A\cos(c\lambda t) + B\sin(c\lambda t) \tag{11}$$

Baby Version: This piece tells us how the trampoline moves as time passes. It tells us that the time part of the problem is a simple movement that smoothly bounces up and down over and over.

6 Spatial ODE

We will now tackle the spatial term in Equation 9, by looking at the radial and angular components and trying to separate this equation into two separate ODEs.

$$\begin{aligned} \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2}\frac{\Theta''}{\Theta} &= -\lambda^2 \\ \frac{r^2}{R}\left(R'' + \frac{1}{r}R'\right) + r^2\lambda^2 &= -\frac{\Theta''}{\Theta} \end{aligned} \tag{12}$$

Once again, we have our left and right hand sides of the equation relying on different variables, which means that they both must equal some constant. Once again, we will conveniently choose a constant, this time m^2.

Why did we choose m^2? We choose a squared variable to make writing our solutions easier, since they will get much larger and we will eventually need to take the square root of our constant. This eliminates lots of easy to mess up square root symbols. This is also a trust the process moment.

$$\frac{r^2}{R}\left(R'' + \frac{1}{r}R'\right) + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = m^2 \tag{13}$$

And just like that, we've severed the final connection, and we are left with two outstanding ODEs to address and solve, like we did with our time ODE in Equation 10.

Baby Version: Now we look at the shape of the trampoline. We take the space part of the problem and break it down in two directions: circles that go around the trampoline (we call this the angular piece) and lines that go from the center to the edge (we call this the radial piece).

7 Angular ODE

Tackling the angular ODE in θ from Equation 13, we see an almost exact replica of the time ODE that we already solved.

$$\Theta'' + m^2\Theta = 0 \tag{14}$$

Look familiar? Our solution matches the form of Equation 11, with a slight exception. Because we are solving this equation on a circle, our solution Θ must be periodic such that $\Theta(\theta) = \Theta(\theta + 2\pi)$, requiring m to be an integer ($m = 0, 1, 2...$).

$$\Theta_m(\theta) = C\cos(m\theta) + D\sin(m\theta) \tag{15}$$

Baby Version: This piece looks at the circles that go around the trampoline, which you can picture as the path you take when you walk around the trampoline edge. Because the trampoline is a full circle, the trampoline surface must perfectly match up when you walk all the way around, which means the waves can only fit into whole numbers like 1, 2, and 3.

8 Radial ODE gives us the Bessel Equation

The final piece of the puzzle is now at hand. We just have to solve the radial ODE. Simple, right? Not quite. Let's see why.

We set up our radial equation and see what it looks like.

$$r^2 R'' + rR' + (\lambda^2 r^2 - m^2)R = 0 \tag{16}$$

Hmmm. This looks different than our previous two ODEs, which means we need to find a different approach to solve it.

Letting $x = \lambda r$, we reformulate Equation 16 as a function of x to get an equation known as the **Bessel Equation**.

If $x = \lambda r$, then we can differentiate x with respect to r such that $\frac{dx}{dr} = \lambda$.

By the chain rule, $\frac{dR}{dr} = \frac{dR}{dx}\frac{dx}{dr} = \frac{dR}{dx} \cdot \lambda$, which gives us the operator identity $\frac{d}{dr} = \lambda \frac{d}{dx}$.

Similarly, $\frac{d^2 R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \left(\lambda \frac{d}{dx}\right)\left(\lambda \frac{dR}{dx}\right) = \lambda^2 \frac{d^2 R}{dx^2}$.

Plugging it all in, we find:

$$x^2 \frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + x\frac{\mathrm{d}R}{\mathrm{d}x} + (x^2 - m^2)R = 0 \tag{17}$$

Baby Version: This piece looks at how the wave moves from the middle of the trampoline to the edge. The math here looks more strange than the problems we solved before, so we give it a special name to help us find the right tools to solve it.

9 Bessel Functions as Solutions of the Bessel Equation

Equation 17 gives us a 2nd order differential equation, and we need to find a solution. By definition, the Bessel equation, written in shorthand as $x^2 y'' + xy' + (x^2 - m^2)y = 0$ will have two linearly independent solutions that we call the Bessel Functions of the First and Second Kind written as $J(x)$ and $Y(x)$.

We can use the Frobenius Method, a technique for finding infinite series solutions to linear second-order differential equations around regular singular points (here where $x = 0$), by assuming a solution of the Bessel Equation given by:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+s} \tag{18}$$

We can take derivatives of this assumed series solution to get y' and y'':

$$y'(x) = \sum_{k=0}^{\infty} (k+s) a_k x^{k+s-1} \tag{19}$$

$$y''(x) = \sum_{k=0}^{\infty} (k+s)(k+s-1) a_k x^{k+s-2} \tag{20}$$

Substituting into the ODE, we will eventually find our solution (see Appendix Section D). In the process of solving for $y(x)$, and in solving a recurrence relation, and by choosing a convenient constant coefficient to clean things up, we will find that:

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{2n+m} \tag{21}$$

We find that the Bessel Function of the First Kind of order m is given by:

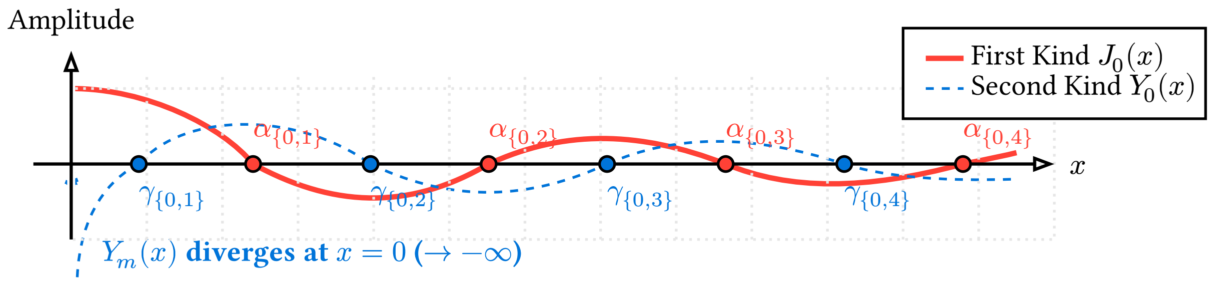
$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+m} \tag{22}$$

where the Gamma function for any positive integer n is given by $\Gamma(n) = (n-1)!$ The Gamma function generalizes the discrete factorial function to a continuous function over the complex numbers, allowing for non-integer values.

More on why we chose to extend our Bessel function to non-integer values and how it relates to our Bessel function of the Second Kind in Appendix Section D.

The general solution to the radial equation, and any second-order linear differential equation, must have two linearly independent solutions. In this case, our solution is given by a linear combination of $J_m(x)$ and $Y_m(x)$ where $x = \lambda r$.

$$R(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r) \tag{23}$$



Since the physical displacement of this membrane is finite and $Y_m(x) \to -\infty$ as $x \to 0$, we can intuitively see that c_2 must be 0. See Appendix Section D for more detail.

Since c_2 is set to 0, we can see the simplified solution to our radial equation is simply the Bessel Function of the First Kind of order m:

$$\begin{aligned} R(r) &= c_1 J_m(\lambda r) + c_2 Y_m(\lambda r) \\ R(r) &= c_1 J_m(\lambda r) \end{aligned} \tag{24}$$

Baby Version: To solve this hard piece, we will use a long chain of numbers that work together to get us closer and closer to the right answer. This answer tells us all about the shape of the wave from the center to the edge, but we have to throw away half of the answer because it says the center of the trampoline stretches all the way into space, which is not true at all!

10 Boundary Conditions Plugged In

The boundary condition from Equation 2 $u(a, \theta, t) = 0$ tells us that the rim of the cylindrical membrane is fixed, and thus the displacement is zero. Plugging this into Equation 24 we get:

$$R(a) = J_m(\lambda a) = 0 \tag{25}$$

For this boundary condition to present a non-trivial solution, λa must be a root of the Bessel function $J_m(x)$. $J_m(x)$ acts like a decaying wave and crosses zero infinitely many times, which means there is an infinite series of roots ($n = 1, 2, 3...$) that will give us infinite radial modes. Letting α_{mn} be the nth positive root of $J_m(x)$, we will see that:

$$\begin{aligned} \lambda_{mn} a &= \alpha_{mn} \\ \lambda_{mn} &= \frac{\alpha_{mn}}{a} \end{aligned} \tag{26}$$

This result produces an index n for our radial modes of the wave equation, though these roots are also dependent on the angular modes m from the Angular ODE. Here, λ_{mn} are the eigenvalues (or the natural vibrational frequencies) of the cylindrical membrane.

It should make intuitive sense that the solution of the wave equation has a dependency on the natural frequencies (or eigenfrequencies) of the trampoline, since they are the rates at which a system will oscillate in the absence of any disturbances.

Baby Version: We remember that the outside edge of the trampoline is held down tightly. Because the edge cannot move, the trampoline is limited in how it can bounce, meaning it can only bounce at certain special speeds, just like instruments can only play certain notes, like C, D, and E.

11 Combine Solutions

Combining the separated solutions we've found across the 3 ODEs we've solved, we can format our combined solution as a sum of all possible integers m for angular modes and n for radial modes.

The general displacement solution $u(r, \theta, t)$ is given by:

$$\begin{aligned} u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)[(A_{mn} \cos m\theta + B_{mn} \sin m\theta) \cos(c\lambda_{mn} t) + \\ (C_{mn} \cos m\theta + D_{mn} \sin m\theta) \sin(c\lambda_{mn} t)] \end{aligned} \tag{27}$$

Baby Version: We take the three small answers we found for time, walking around the edge, and the distance from the center, and we put them back together. We add up all the ways the trampoline can bounce into one big equation.

12 Solution Coefficients

Given the initial conditions from Equation 3 $u(r, \theta, 0) = f(r, \theta)$ and $u_t(r, \theta, 0) = g(r, \theta)$ and the orthogonality conditions below, we can solve for the initial position coefficients.

$$\begin{aligned} \text{Angular: } & \int_0^{2\pi} \cos m\theta \cos p\theta d\theta = \pi \delta_{mp} \\ \text{Radial: } & \int_0^a J_m(\lambda_{mn} r) J_m(\lambda_{mp} r) r dr = \frac{a^2}{2} [J_{m+1}(\alpha_{mn})]^2 \delta_{np} \end{aligned} \tag{28}$$

Substituting into Equation 27 at $t = 0$, we find that:

$$f(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(A_{mn} \cos m\theta + B_{mn} \sin m\theta) \tag{29}$$

We find our coefficients A_{mn} and B_{mn} by integrating over the disk bounded by $0 < r < a$ and $0 < \theta < 2\pi$ to show that:

$$A_{mn} = \frac{\int_0^a \int_0^{2\pi} f(r,\theta) J_m(\lambda_{mn} r) \cos m\theta r d\theta dr}{\frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \tag{30}$$

where $\varepsilon_m = 1$ when $m = 0$, and 2 if $m > 0$. See Appendix Section E for more detail. Similarly, we find that:

$$B_{mn} = \frac{\int_0^a \int_0^{2\pi} f(r,\theta) J_m(\lambda_{mn} r) \sin m\theta r d\theta dr}{\frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \tag{31}$$

This is a little tedious, but we're almost there!

To find our initial velocity coefficients, we can differentiate Equation 27 $u_t(r, \theta, t = 0)$ with respect to t to get:

$$g(r,\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c\lambda_{mn} J_m(\lambda_{mn} r)(C_{mn} \cos m\theta + D_{mn} \sin m\theta) \tag{32}$$

Solving for our final two coefficients C_{mn} and D_{mn}, we get:

$$C_{mn} = \frac{\int_0^a \int_0^{2\pi} g(r,\theta) J_m(\lambda_{mn} r) \cos m\theta r d\theta dr}{c\lambda_{mn} \frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \tag{33}$$

$$D_{mn} = \frac{\int_0^a \int_0^{2\pi} g(r,\theta) J_m(\lambda_{mn} r) \sin m\theta r d\theta dr}{c\lambda_{mn} \frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \tag{34}$$

Remembering that the eigenvalues λ_{mn} are given by the roots of the Bessel function $\lambda_{mn} = \frac{\alpha_{mn}}{a}$, we can plug in our coefficients to find the full solution:

$$\begin{aligned} u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r) \Bigg[\Bigg(&\frac{\int_0^a \int_0^{2\pi} f(r,\theta) J_m(\lambda_{mn} r) \cos m\theta r d\theta dr}{\frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \cos m\theta + \\ &\frac{\int_0^a \int_0^{2\pi} f(r,\theta) J_m(\lambda_{mn} r) \sin m\theta r d\theta dr}{\frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \sin m\theta \Bigg) \cos(c\lambda_{mn} t) + \\ &\Bigg(\frac{\int_0^a \int_0^{2\pi} g(r,\theta) J_m(\lambda_{mn} r) \cos m\theta r d\theta dr}{c\lambda_{mn} \frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \cos m\theta + \\ &\frac{\int_0^a \int_0^{2\pi} g(r,\theta) J_m(\lambda_{mn} r) \sin m\theta r d\theta dr}{c\lambda_{mn} \frac{a^2\pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn})\right]^2} \sin m\theta \Bigg) \sin(c\lambda_{mn} t) \Bigg] \end{aligned} \tag{35}$$

And just like that, we've found the complete analytical solution to the wave equation on a circular membrane! You did it! Pat on the back.

Baby Version: Finally, we use some math tricks and our starting shape and starting speed to find the last missing numbers, and then we are completely done. Yay baby!

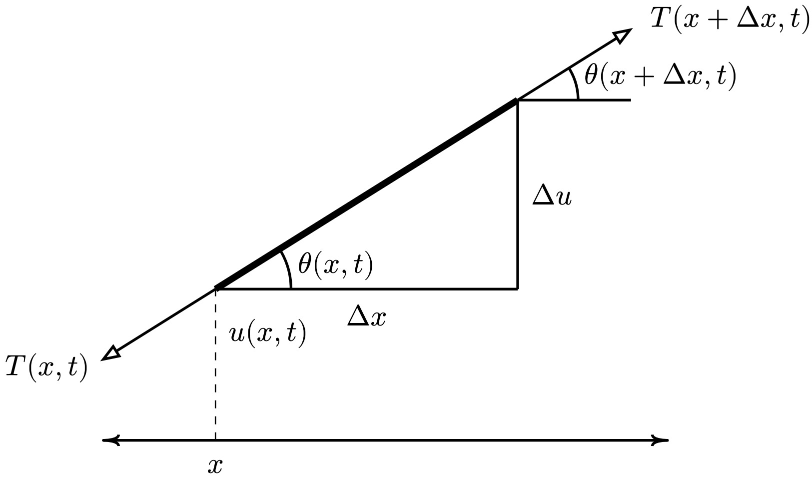
Appendix

A Derivation of the Wave Equation

We will apply Newton's law ($\vec{F} = m\vec{a}$) to small transverse vibrations of an elastic string to find the wave equation, and then we can easily extend this principle to higher-order dimensions, such as a vibrating membrane.

Consider a tiny section of a string with the following properties:

- $u(x,t) =$ vertical displacement of the string from the axis at position x and time t.
- $\theta(x,t) =$ angle between the string and a horizontal line at position x and time t.
- $T(x,t) =$ tension in the string at position x and time t.
- $\rho(x) =$ linear density ($\frac{m}{l}$) of the string at position x.



A1 Force Balance and Newton's Law

The forces acting on the tiny section of string are:

1. Tension to the right, which has magnitude $T(x + \Delta x, t)$ and acts at an angle $\theta(x + \Delta x, t)$ above the horizontal.
2. Tension to the left, which has magnitude $T(x,t)$ and acts at an angle $\theta(x,t)$ below the horizontal.
3. Negligible external forces, like gravity, relative to the tension in the string.

The mass of the element of string is $\rho(x)\sqrt{(\Delta x)^2 + (\Delta u)^2}$. The vertical component of Newton's Second Law ($\Sigma \vec{F_y} = m\vec{a}$) states:

$$\rho(x)\sqrt{(\Delta x)^2 + (\Delta u)^2}\frac{\partial^2 u}{\partial t^2}(x,t) = T(x + \Delta x, t)\sin\theta(x + \Delta x, t) - T(x,t)\sin\theta(x,t) \tag{36}$$

.

We can pull out Δx from the square root (leaving us with $1 + \frac{\Delta u}{\Delta x}$), allowing us to divide both sides by Δx and taking the limit as $\Delta x \to 0$ to give us the partial derivatives in the following equation:

$$\rho(x)\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}[T(x,t)\sin\theta(x,t)] \tag{37}$$

.

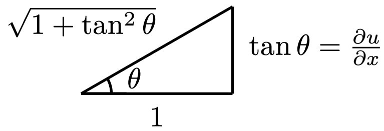
Expanding the derivative on the right side with the product and chain rule:

$$\rho(x)\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial T}{\partial x}(x,t)\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\frac{\partial \theta}{\partial x}(x,t) \tag{38}$$

.

From the geometry of the string, we know that $\tan\theta(x,t) = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}(x,t)$. We want to replace the $\sin\theta$ and $\cos\theta$ terms above with $\frac{\partial u}{\partial x}$. To do so, we can construct a reference triangle to leverage SOH CAH TOA and reframe our trigonometric functions in terms of $\tan\theta$ and thus $\frac{\partial u}{\partial x}$.

We construct a right triangle that satisfies the TOA property (tan θ = opposite over adjacent) using the same θ as our string. Letting the adjacent side = 1, our opposite side is given by $\tan\theta = \frac{\text{opposite}}{1} \to \text{opposite} = \tan\theta$. Using the Pythagorean theorem, we find that the hypotenuse is $\sqrt{1+\tan^2\theta}$.



SOH (sin is opposite over hypotenuse) CAH (cosine is adjacent over hypotenuse) implies the following trigonometric relationships that allow us to reframe sin and cos in terms of $\frac{\partial u}{\partial x}$:

$$\sin\theta(x,t) = \frac{\frac{\partial u}{\partial x}(x,t)}{\sqrt{1+\left(\frac{\partial u}{\partial x}(x,t)\right)^2}} \tag{39}$$

$$\cos\theta(x,t) = \frac{1}{\sqrt{1+\left(\frac{\partial u}{\partial x}(x,t)\right)^2}} \tag{40}$$

$$\theta(x,t) = \tan^{-1}\frac{\partial u}{\partial x}(x,t) \tag{41}$$

$$\frac{\partial \theta}{\partial x}(x,t) = \frac{\frac{\partial^2 u}{\partial x^2}(x,t)}{1+\left(\frac{\partial u}{\partial x}(x,t)\right)^2} \tag{42}$$

.

A2 Small Angle Approximation

Substituting these formulas into our force equation gives a mess. However, we can simplify considerably by looking only at small vibrations. A small vibration (or small angle) approximation lets us assume that $|\theta(x,t)| \ll 1$ for all x and t.

This implies that $|\tan\theta(x,t)| \ll 1$, meaning $|\frac{\partial u}{\partial x}(x,t)| \ll 1$. Therefore, we can make the following small angle approximations:

$$\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} \approx 1 \tag{43}$$

$$\sin\theta(x,t) \approx \frac{\partial u}{\partial x}(x,t) \tag{44}$$

$$\cos\theta(x,t) \approx 1 \tag{45}$$

$$\frac{\partial \theta}{\partial x}(x,t) \approx \frac{\partial^2 u}{\partial x^2}(x,t) \tag{46}$$

Substituting these into the vertical component equation gives:

$$\rho(x)\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial T}{\partial x}(x,t)\frac{\partial u}{\partial x}(x,t) + T(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) \tag{47}$$

.

A3 The Horizontal Component

This is one equation in the two unknowns u and T. We use the horizontal component of Newton's law of motion to proceed. As a second simplification, we assume that there are only transverse vibrations. Since the tiny string element moves only vertically, the net horizontal force on it must be zero:

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0 \tag{48}$$

.

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives:

$$\frac{\partial}{\partial x}[T(x,t)\cos\theta(x,t)] = 0 \tag{49}$$

.

For small amplitude vibrations, $\cos\theta$ is very close to one, so the equation above tells us that $\frac{\partial T}{\partial x}(x,t) \approx 0$. This implies that T is a function of t only, not x.

A4 Final Wave Equation

For small, transverse vibrations, our primary equation simplifies further to:

$$\rho(x)\frac{\partial^2 u}{\partial t^2}(x,t) = T(t)\frac{\partial^2 u}{\partial x^2}(x,t) \tag{50}$$

We're in the home stretch! Assuming the string density ρ is constant and independent of x and that tension T is independent of t, we get:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) \tag{51}$$

where $c = \sqrt{\frac{T}{\rho}}$.

Extending to multiple dimensions where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \ldots$, we see that:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{52}$$

which we know as the wave equation.

B Polar Laplacian

We must transform $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ into polar coordinates (r, θ).

Coordinate Transformations are given by:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2+y^2} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right)
\end{aligned}
\tag{53}
$$

Using the chain rule for the function tree represented by $u = u(r, \theta)$ and $r = r(x, y)$ and $\theta = \theta(x, y)$ given by:

$$
\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}
\tag{54}
$$

We can find the individual derivatives in Equation 54:

$$
\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{r} = \cos\theta \\
\frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r} \\
\frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta \\
\frac{\partial \theta}{\partial x} &= \frac{\cos\theta}{r}
\end{aligned}
\tag{55}
$$

Thus:

$$
\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}
\tag{56}
$$

We can find the second derivative by applying the operator to itself:

$$
\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)
\tag{57}
$$

We expand to find:

$$
\frac{\partial^2}{\partial x^2} = \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\frac{\partial}{\partial r}\frac{\sin\theta}{r}\frac{\partial}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}
\tag{58}
$$

Similarly for y, we can find the first and second derivative.

$$
\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}
\tag{59}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) \\
\frac{\partial^2}{\partial y^2} &= \sin^2\frac{\partial^2}{\partial r^2} + \sin\theta\frac{\partial}{\partial r}\frac{\cos\theta}{r}\frac{\partial}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}
\end{aligned}
\tag{60}
$$

Now that we have $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ in terms of r and θ, we can plug them in:

$$
\nabla_p^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\tag{61}
$$

$$\nabla_p^2 = \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\frac{\partial}{\partial r}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \sin^2\frac{\partial^2}{\partial r^2} + \sin\theta\frac{\partial}{\partial r}\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \tag{62}$$

Remembering that $(u \cdot v)' = u' \cdot v + u \cdot v'$, which applies to the last term of $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$, we can find that:

$$\nabla_p^2 = \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} - \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{63}$$

Canceling terms out because $\sin^2\theta + \cos^2\theta = 1$, we get:

$$\nabla_p^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \tag{64}$$

Plugging the polar Laplacian into the wave equation, we find the wave equation in polar coordinates as:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \tag{65}$$

C Time and Angular ODEs

Solving 2nd Order Linear Homogenous Differential Equations with Constant Coefficients

Given the following ODE, how do we find the solution?

$$T'' + c^2\lambda^2 T = 0 \tag{66}$$

Use an **ansatz**, or an educated guess of the functional form of the solution, to reduce the ODE into solvable algebraic equations. For linear homogenous ODEs with constant coefficients, we will typically use an exponential ansatz of the form:

$$T(t) = e^{rt} \tag{67}$$

Plugging our ansatz into our ODE, we get:

$$r^2 e^{rt} + c^2\lambda^2 e^{rt} = 0 \tag{68}$$

Since e^{rt} cannot be 0, we find that:

$$\begin{aligned} r^2 + c^2\lambda^2 &= 0 \\ r^2 + c^2\lambda^2 &= 0 \\ r^2 &= -c^2\lambda^2 \\ r &= \pm ic\lambda \end{aligned} \tag{69}$$

Now that we have identified r, we can plug in our ansatz as a superposition of our two roots:

$$T(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \tag{70}$$

Based on Euler's Formula:

$$\begin{aligned} e^{ix} &= \cos(x) + i\sin(x) \\ e^{-ix} &= \cos(x) - i\sin(x) \end{aligned} \tag{71}$$

You may have seen this in a different form when $x = \pi$ given by:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 \tag{72}$$

We can then substitute:

$$\begin{aligned} T(t) &= c_1[\cos(\omega t) + i\sin(\omega t)] + c_2[\cos(\omega t) - i\sin(\omega t)] \\ &= (c_1 + c_2)\cos(\omega t) + i(c_1 - c_2)\sin(\omega t) \\ &\text{letting} \quad A = c_1 + c_2, B = i(c_1 - c_2) \\ &= A\cos(\omega t) + B\sin(\omega t) \end{aligned} \tag{73}$$

We now have our general form of the solution for a 2nd order linear homogenous differential equation with constant coefficients, given by a sum of trigonometric functions. How delightful!

D Solving Bessel's Equation

We've already seen the set up for how to solve the Bessel Equation with the Frobenius Method:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+s} \tag{74}$$

We can take the derivative of this assumed series solution form to get y' and y'':

$$y'(x) = \sum_{k=0}^{\infty} (k+s) a_k x^{k+s-1} \tag{75}$$

$$y''(x) = \sum_{k=0}^{\infty} (k+s)(k+s-1) a_k x^{k+s-2} \tag{76}$$

Substituting the above into the Bessel Equation given by:

$$x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - m^2)R = 0 \tag{77}$$

We find that:

$$\begin{gathered} \sum_{k=0}^{\infty} (k+s)(k+s-1) a_k x^{k+s} + \sum_{k=0}^{\infty} (k+s) a_k x^{k+s} + \\ \sum_{k=0}^{\infty} a_k x^{k+s+2} - m^2 \sum_{k=0}^{\infty} a_k x^{k+s} = 0 \end{gathered} \tag{78}$$

We can group terms to get:

$$\sum_{k=0}^{\infty} [k^2 + 2ks + s^2 - k - s + k + s - m^2] a_k x^{k+s} + \sum_{k=0}^{\infty} a_k x^{k+s+2} = 0 \tag{79}$$

Simplifying this equation, we get:

$$\sum_{k=0}^{\infty} [(k+s)^2 - m^2] a_k x^{k+s} + \sum_{k=0}^{\infty} a_k x^{k+s+2} = 0 \tag{80}$$

We align the infinite sums by shifting the index so that they both are in terms of x^{k+s}:

$$\sum_{k=0}^{\infty}[(k+s)^2 - m^2]a_k x^{k+s} + \sum_{k=2}^{\infty} a_{k-2}x^{k+s} = 0 \tag{81}$$

Now that we are using the same exponent basis for our power series, we can shift the index of the first infinite sum to start at $k = 2$ by manually popping out the first two terms where $k = 0, 1$.

$$(s^2 - m^2)a_0 x^s + [(1+s)^2 - m^2]a_1 x^{s+1} + \sum_{k=2}^{\infty}([(k+s)^2 - m^2]a_k + a_{k-2})x^{k+s} = 0 \tag{82}$$

For the equation to be zero for all x, the coefficient of every power of x must be 0. This allows us to tackle coefficients separately, starting with x^s.

$$\begin{gathered}(s^2 - m^2)a_0 = 0 \\ s = \pm m\end{gathered} \tag{83}$$

We choose positive values of s because $x^{-m} \to \infty$ when $x = 0$, and our solution is finite at $x = 0$, so $s = +m$.

Looking at x^{s+1}, we find that:

$$[(1+m)^2 - m^2]a_1 = 0 \to (1+2m)a_1 = 0 \to a_1 = 0 \text{ since } m \geq 0 \tag{84}$$

Lastly, we can look at x^{k+s} to get a recurrence relation:

$$\begin{gathered}[(k+m)^2 - m^2]a_k + a_{k-2} = 0 \\ (k^2 + 2mk)a_k = -a_{k-2} \\ a_k = \frac{-a_{k-2}}{k(k+2m)}\end{gathered} \tag{85}$$

Since $a_1 = 0$, all odd coefficients a_3, a_5, a_7...are zero. This means we can let $k = 2n$ since our coefficients will only have a value for even integers.

$$a_{2n} = \frac{-a_{2n-2}}{2n(2n+2m)} = \frac{-a_{2n-2}}{2^2 n(n+m)} \tag{86}$$

By checking the values of $n = 1, 2, 3...$ we can see that the general formula will be:

$$a_{2n} = \frac{(-1)^n}{2^{2n}n!(m+1)(m+2)...(m+n)}a_0 = \frac{(-1)^n(m)!}{2^{2n}n!(m+n)!}a_0 \tag{87}$$

We can choose the value of a_0, so for convenience, we will choose a value that simplifies our expression:

$$a_0 = \frac{1}{2^m m!} \tag{88}$$

Plugging a_0 in, we find that our solution is the Bessel Function of the First Kind of order m:

$$\begin{gathered}y(x) = \sum_{k=0}^{\infty} a_k x^{k+s} = \sum_{n=0}^{\infty} a_{2n}x^{2n+m} \to \\ y(x) = J_m(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+m)!}\left(\frac{x}{2}\right)^{2n+m}\end{gathered} \tag{89}$$

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+m} \tag{90}$$

where the Gamma function for any positive integer n is given by $\Gamma(n) = (n-1)!$ The Gamma function generalizes the discrete factorial function to a continuous function over the complex numbers, allowing for non-integer values.

Any second-order linear ODE must have exactly two linearly independent solutions to form a complete general solution. We know a second solution exists such that the general solution is of the form:

$$R(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r) \tag{91}$$

Using the Frobenius method, we saw that we had two roots for the power series exponent: $s = +m$ and $s = -m$. When m is an integer (as it is defined here by the angular ODE), we see that the resulting second series is a direct multiple of the first series such that:

$$J_{-m} = (-1)^m J_m(x) \tag{92}$$

This is why the Frobenius method fails to provide a second independent solution. Because J_{-m} is linearly dependent on $J_m(x)$, we still only have one unique solution for integer values of m. What a tragedy!

When the roots of our indicial equation differ by an integer value (here it is $m - (-m) = 2m$), Frobenius theory dictates that the second linearly independent solution takes a specific form of:

$$Y_m(x) = C J_m(x) \ln(x) + x^{-m} \sum_{k=0}^{\infty} b_k x^k \tag{93}$$

This form comes from the reduction of order technique to find a second solution from a first solution. Solving this equation involves plugging $Y_m(x)$ back into the Bessel Equation and gets very messy very quickly.

To make things simpler, we typically define the Bessel Function of the Second Kind $Y_m(x)$ with the Weber definition. We do this because non-integers provide linearly independent solutions for $J_m(x)$ and $Y_m(x)$.

The Weber definition tells us that the limit of the linearly independent, non-integer case where ν is a non-integer and m is an integer, gives us our second solution:

$$Y_m(x) = \lim_{\nu \to m} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{94}$$

Evaluating this equation creates an indeterminate form of $\frac{0}{0}$ as $\nu \to m$. L'Hopital's rule tells us that we must differentiate the numerator and the denominator with respect to ν.

Remember when we decided to use the gamma function Γ, the continuous extension of the factorial, in our Bessel function definition even though our integer m did not necessitate it? It was for precisely this reason: so that we could have a continuous and differentiable function with respect to its order $J_\nu(x)$ to help us find our second solution!

After differentiating, the expanded form of the Bessel Function of the Second Kind $Y_m(x)$ is:

$$
\begin{aligned}
Y_m(x) = \frac{2}{\pi} J_m(x)\left(\ln\left(\frac{x}{2}\right) + \gamma\right) - \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-m} - \\
\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(m+k)!} \left(\frac{x}{2}\right)^{2k+m} (H_k + H_{k+m})
\end{aligned}
\tag{95}
$$

where H_k is the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ and γ is the Euler-Mascheroni constant (≈ 0.5772), which arises when taking the derivative of the gamma function Γ in $J_m(x)$.

The first two terms of the Bessel Function of the Second Kind $Y_m(x)$ both blow up to $-\infty$ when $x \to 0$ (we see that $\ln(0) \to -\infty$ and $(m-1)! \frac{0}{2}^{-m} \to \infty$), and we know that membrane displacement is finite, so we typically immediately set the coefficient c_2 to 0 to prevent this process and thus short circuit a difficult piece of math with our physical intuition.

This is how we get to the solution of the Bessel Equation and thus our radial ODE:

$$
R(r) = c_1 J_m(\lambda r) \tag{96}
$$

E Using Orthogonality Conditions to find coefficients

We will walk through the steps to find A_{mn} to see how we use the orthogonality conditions to obtain a specific coefficient.

$$
\begin{aligned}
&\text{Angular: } \int_0^{2\pi} \cos m\theta \cos p\theta d\theta = \pi \delta_{mp} \\
&\text{Radial: } \int_0^{a} J_m(\lambda_{mn} r) J_m(\lambda_{mp} r) r dr = \frac{a^2}{2} [J_{m+1}(\alpha_{mn})]^2 \delta_{np}
\end{aligned}
\tag{97}
$$

where the the Kronecker delta δ_{mp} is defined as 1 for $m = p$ and 0 for $m \neq p$. We use the Kronecker delta on discrete sets like m and p indices, as opposed to the Dirac delta $\delta(x)$ for continuous variables.

The Angular Orthogonality condition checks if two angular waves with frequencies m and p are orthogonal. The Radial Orthogonality condition states that two Bessel functions of the same order m but different radial modes, or roots, n and p are orthogonal with respect to the weight r.

We will start by isolating the angular order m of our initial condition for $f(r, \theta)$. First we multiply both sides by $\cos(p\theta)$ and integrate with $d\theta$.

$$
f(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(A_{mn} \cos m\theta + B_{mn} \sin m\theta) \tag{98}
$$

$$
\begin{aligned}
\int_0^{2\pi} f(r, \theta) \cos p\theta d\theta = \sum_{m,n}^{\infty} J_m(\lambda_{mn} r)\Bigg[\Bigg(A_{mn} \int_0^{2\pi} \cos m\theta \cos p\theta d\theta + \\
B_{mn} \int_0^{2\pi} \sin m\theta \cos p\theta d\theta \Bigg]
\end{aligned}
\tag{99}
$$

We notice a few things with the help of our angular orthogonality property: the $\sin \cdot \cos$ integral is always 0 and the $\cos \cdot \cos$ integral is only non-zero when $m = p$. The general integral given by I simplifies in two cases when $m = p$:

$$I = \int_0^{2\pi} \cos m\theta \cos p\theta d\theta$$

$$I = \int_0^{2\pi} \cos^2(m\theta) d\theta = \int_0^{2\pi} \frac{1 + \cos(2m\theta)}{2} d\theta = \pi \text{ when } m = p, m > 0 \tag{100}$$

$$I = \int_0^{2\pi} (1)^2 d\theta = 2\pi \text{ when } m = p = 0$$

This generalizes such that:

$$\begin{gathered} I = \int_0^{2\pi} \cos^2(m\theta) d\theta = \frac{2\pi}{\varepsilon_m} \text{ where} \\ \text{if } m = 0, \varepsilon_m = 1 \to I = 2\pi \\ \text{if } m > 0, \varepsilon_m = 2 \to I = \pi \end{gathered} \tag{101}$$

We are left with:

$$\int_0^{2\pi} f(r, \theta) \cos p\theta d\theta = \frac{2\pi}{\varepsilon_p} \sum_{n=1}^{\infty} A_{pn} J_p(\lambda_{pn} r) \tag{102}$$

Now we need to build on Equation 102 and isolate the radial mode n by multiplying both sides by J_p and the integration weight r and integrating over the radius $0 < r < a$.

$$\begin{gathered} \int_0^a \left[\int_0^{2\pi} f(r, \theta) \cos p\theta d\theta \right] J_p(\lambda_{pq} r) r dr) = \\ \left(\frac{2\pi}{\varepsilon_p} \right) \sum_{n=1}^{\infty} A_{pn} \int_0^a J_p(\lambda_{pn} r) J_p(\lambda_{pq} r) r dr \end{gathered} \tag{103}$$

By the radial orthogonality property, the integral on the right is zero unless $n = q$. When $n = q$, the integral evaluates to:

$$\int_0^a J_p(\lambda_{pn} r) J_p(\lambda_{pq} r) r dr = \frac{a^2}{2} \left[J_{p+1}(\alpha_{pq}) \right]^2 \tag{104}$$

We can now solve for $A_{pn} = A_{pq}$ and swap out p, q for our more familiar m, n, giving us:

$$A_{mn} = \frac{\int_0^a \int_0^{2\pi} f(r, \theta) J_m(\lambda_{mn} r) \cos m\theta r d\theta dr}{\frac{a^2 \pi}{\varepsilon_m} \left[J_{m+1}(\alpha_{mn}) \right]^2} \tag{105}$$