On the Logarithmic Decrement, the Kelvin-Voigt Model, and Breast Pain

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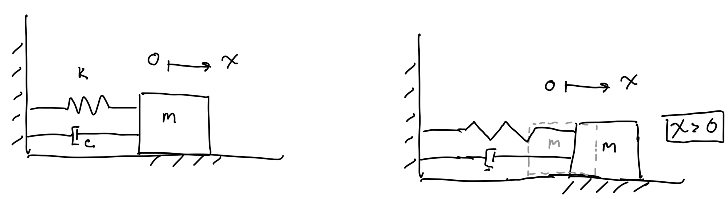
1 Introduction

In this paper, we will explore what the logarithmic decrement means, how it is derived, and what purposes it might serve. We will use a simplified model to understand the basic principles of vibration, but also because more complex models are far too difficult. Keep in mind, however, that the same principles underlie all vibrations. I'd encourage you to grab a piece of paper (a big one) and follow along by drawing the diagrams and trying to do some of the math yourself! It's fun!

Our objective: considering a simple system of a mass, a damper, and a spring, determine the damping ratio ζ using the method of the logarithmic decrement. We will use this damping ratio to understand the principles of vibration, the decay of oscillation, and viscoelastic material models.

2 A harmonic oscillator walks into a bar

Our story begins, as most textbook-style stories do, considering a mass m on the ground. This mass is nothing special, and were it not the protagonist of this story, this mass would probably stay stationary on the ground, left to waste away amidst the crippling loneliness of inertia. But such is not the case, as this mass is excited to see. For we, the meddling audience, have connected the mass m to a vertical wall by attaching to it a spring of spring constant k and a viscous damper, or dashpot, of damping coefficient c. We'll call this system a harmonic oscillator, and as it can only move left or right, it has but 1 degree of freedom.



(a) A single degree of freedom harmonic oscillator, or SDOFHO for confusing and short.

(b) A displaced SDOFHO where we can see that $x > 0$

Figure 1: Our SDOFHO system

To those unfamiliar with the dashpot, just know that we will use it to quantify the effects of viscous friction that steal energy from our system, proportional to velocity but in the opposite direction. The spring also provides an opposing force proportional to the distance it is stretched, so it will resist compression or stretching. We can set a local coordinate system, named x, at the center of our mass, and displace our system such that the mass moves away from the wall with a positive displacement and positive velocity (going right), meaning $x > 0$

and $\dot{x} > 0$. There is no external force acting on this system to make it oscillate, so we say that our system is under *free vibration*.

Let's construct a free body diagram of the mass, noting all forces acting on it and excluding gravity (though it doesn't really matter here, because gravity doesn't act in the same direction as our movement). Note that the damping effect of the dashpot is taken to be in the same direction as the force exerted by the spring, as the spring acts to impede any displacement, and the dashpot seeks to dampen velocity. The spring exerts a force equal to kx, the product of the spring constant and the displacement, while the dashpot exerts a force equal to $c\dot{x}$, the product of the damping constant and the velocity.

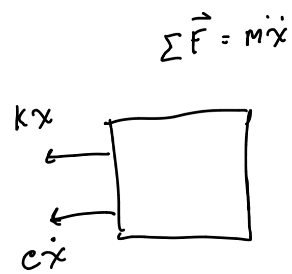


Figure 2: FBD of the mass

Using this free body diagram and Newton's Second Law of motion, we find an ordinary differential equation that says all the forces acting on the mass are equal to the mass times the second derivative of position ($\ddot{x}$, or acceleration).

$$m\ddot{x} = -kx - c\dot{x} \tag{1}$$

We can define several variables in the meanwhile to help us turn this ODE into something more useful. We'll define a dimensionless damping ratio ζ and a natural frequency, or eigenfrequency, ω_n:

$$\zeta = \frac{c}{2\sqrt{mk}}$$
$$\omega_n = \sqrt{\frac{k}{m}}$$

The damping ratio ζ is key to our understanding of vibrations, and it characterizes the dissipation of oscillation. In other words, how readily does our system stop vibrating? A system can be undamped ($\zeta = 0$), underdamped ($\zeta < 1$), critically damped ($\zeta = 1$), or overdamped ($\zeta > 1$).

We now see that equation [1] can be converted to the following, the **Damped Free Vibration Equation**, or the equation of a system that has viscous damping acting on it, but no driving force, with simple algebraic manipulation. Try it yourself starting here and working your way back to equation [1].

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 = 0 \tag{2}$$

3 Eine kleine Nachtmathe

Great! We've found an equation (a 2nd order ordinary differential equation, to be exact) that describes our SDOFHO (single degree of freedom harmonic oscillator), but to what end? We can't do much with this nice equation, as compact as it is. We should solve it to find out what it says about $x(t)$, or the position of the mass. That is much more useful to us than a general form that combines velocity $\dot{x}$ and acceleration $\ddot{x}$.

We solve equation [2] by guessing a form of the solution $x(t) = Ae^{st}$, and plugging it into [2] to find the roots $s_{1,2}$. Several sample steps are shown below, but the majority of it is straightforward.

$$\begin{aligned} x &= Ae^{st} \\ \dot{x} &= sAe^{st} \\ \ddot{x} &= s^2Ae^{st} \end{aligned}$$

As follows: we can plug the above into Equation [2] and simplify:

$$\begin{aligned} s^2Ae^{st} + 2\zeta\omega_n sAe^{st} + \omega_n^2 Ae^{st} &= 0 \\ (s^2 + 2\zeta\omega_n s + \omega_n^2)Ae^{st} &= 0 \\ s^2 + 2\zeta\omega_n s + \omega_n^2 &= 0 \end{aligned}$$

Solve the *quadratic equation* above to find the roots s_1 and s_2, keeping in mind that for now, we will assume that the system is underdamped, so $\zeta < 1$. This equation has two roots, being a second order ODE, and we find them to be:

$$\begin{aligned} s_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \\ s_1 &= -\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2} \\ s_2 &= -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \end{aligned}$$

To simplify, as we are prone to doing, we can consider the damped natural frequency ω_d to be the $\omega_n\sqrt{1-\zeta^2}$ term in our roots above.

$$\begin{aligned} s_1 &= -\zeta\omega_n - i\omega_d \\ s_2 &= -\zeta\omega_n + i\omega_d \end{aligned}$$

Plugging our roots back into our sample solution $x(t) = Ae^{st}$, we can see that the solution will be some *linear combination* of our two different roots, so we represent our sample solution as:

$$x(t) = Ae^{s_1 t} + Be^{s_2 t} \tag{3}$$

$$x(t) = Ae^{(-\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2})t} + Be^{(-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2})t} \tag{4}$$

$$x(t) = Ae^{(-\zeta\omega_n - i\omega_d)t} + Be^{(-\zeta\omega_n + i\omega_d)t} \tag{5}$$

This is not only an ugly equation, it's useless to those of us that don't get what i is supposed to mean in the context of vibrations. We'll use two incredibly cool tricks that will make our lives easier. First, Euler's formula [6]. This will be used when solving 99% of all ODEs, as it can convert imaginary numbers to real ones. How cool is that? Second, we will use a nifty trig identity [7] to make our problem even simpler.

$$e^{i\theta} = cos\theta + isin\theta \tag{6}$$

$$Acos\theta + Bsin\theta = \sqrt{A^2 + B^2}cos(\theta - \phi) \tag{7}$$

$$\phi = \arctan(\frac{A}{B}) \tag{8}$$

It's good to know that with some tinkering, we can convert all exponentials raised to imaginary numbers to linear combinations of sine and cosine functions, but this math does rely on some expanding (which you should do on your big piece of paper). In general, remember that whenever we see the function below, we will end up with:

$$Ae^{ix} + Be^{-ix} = Ccos(x) + Dsin(x)$$

C should be some arbitrary constant equal to $(A+B)$ and D should be equal to $(A - B)i$. We can perform this expansion easily if we know that:

$$cos(-\theta) = cos(\theta)$$
$$sin(-\theta) = -sin(\theta)$$

We are left with the following term for $x(t)$. At this point, it doesn't look too bad because we don't have any imaginary terms, but we can simplify even further yet! We'll use that nifty trig identity on Equation [5], then convert the square root term to a general amplitude X.

$$x(t) = e^{-\zeta\omega_n t}(Ae^{-i\omega_d t} + Be^{i\omega_d t}) \tag{9}$$

$$x(t) = e^{-\zeta\omega_n t}[Ccos(\omega_d t) + Dsin(\omega_d t)] \tag{10}$$

$$x(t) = e^{-\zeta\omega_n t}[\sqrt{C^2 + D^2}cos(\omega_d t - \phi)] \tag{11}$$

$$x(t) = Xe^{-\zeta\omega_n t}cos(\omega_d t - \phi) \tag{12}$$

This beautiful equation says that the position of our mass m is simply the product of some amplitude X damped (or made smaller) by a $e^{-\zeta\omega_n t}$ term and a cosine function that acts as a phase shift.

Keep in mind that when using all the relevant substitutions, this formula is quite complex. Should the fancy arise, we could use the initial conditions in [10], plugging them in to to find the fully expanded expression:

$$x(0) = x_0 = C \tag{13}$$
$$\dot{x}(0) = v_0 = -\zeta\omega_n C + D\omega_d \tag{14}$$

Thus:

$$X = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \tag{15}$$
$$\phi = \arctan\left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d x_0}\right) \tag{16}$$

And lest we forget (as it has been a while):

$$\omega_n = \sqrt{\frac{k}{m}} \tag{17}$$
$$\zeta = \frac{c}{2\sqrt{mk}} \tag{18}$$

Now this is the point where things get interesting. We have derived a very nice function that can tell us the position of our mass at any time, given its initial position, initial velocity, the time that has elapsed, the damping ratio, the spring constant, and the mass $(x_0, v_0, t, \zeta, k, m)$. Phew, that's a lot of variables, eh?

4 Logarithmic decrement: a user's guide

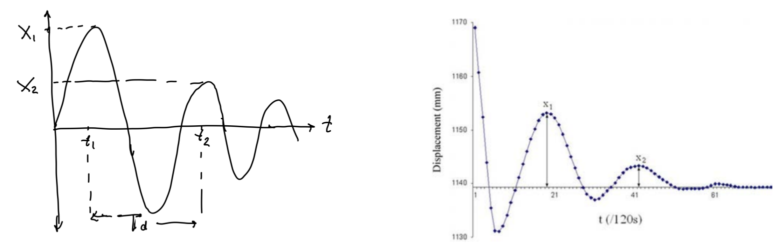
At this point, we will introduce the Logarithmic Decrement. For our purposes, it is much easier to measure the position $x(t)$ than it is to find the damping ratio. Accordingly, let's construct a scenario that will allow us to use $x(t)$ to find ζ!

Here's a general procedure for how to use the Logarithmic Decrement, or the LogDec in some circles.

1. Obtain a time versus amplitude plot of a damped, freely vibrating mass. It will look like a roller coaster with a very big hill at the beginning and smaller hills near the end.

2. Take the natural log of the ratio of the first two, or any two, peak amplitudes that we can see in Figure [3].

3. Fiddle around with the equations until we can isolate ζ as a function of things we know.
4. We're done!



(a) Sample displacement versus time graph

(b) Real data taken from an experiment of displacement versus time from Chen et al.

Figure 3: Time domain plots used in the Logarithmic Decrement

This is fairly nice, as we don't need to mess with the very complex parts of Equation [5]. We already know the amplitudes, as measured experimentally, so the procedure is straightforward. Here are some things to know before we start proceeding through the LogDec:

$$\delta = ln\left(\frac{H_1}{H_2}\right)$$

H is taken to be the damped amplitudes of the peaks of the graph, which are pictured as X_1 and X_2. We will assume that the time elapsed between the two peaks is the damped period, or T_d. We can write this as a function of the damped natural frequency ($f = \omega_d = 2\pi/T_d$):

$$t_2 = t_1 + T_d$$
$$t_2 = t_1 + \frac{2\pi}{\omega_d}$$

Don't forget that we defined the damped natural frequency ω_d as $\omega_n\sqrt{1-\zeta^2}$, a function of the natural frequency and the damping ratio.

Moving along, we'll plug in the amplitudes H_1 and H_2 of our equations of motion and see what we can find. Remember that the damped amplitude of our equation of motion is given by $Xe^{-\zeta\omega_n t}$.

$$\begin{aligned}\frac{H_1}{H_2} &= \frac{Xe^{-\zeta\omega_n t_1}}{Xe^{-\zeta\omega_n t_2}} \\ &= \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n\left(t_1+\frac{2\pi}{\omega_d}\right)}} \\ &= e^{\zeta\omega_n \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}} \\ &= e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}\end{aligned}$$

Upon taking the natural log, we are left with:

$$\delta = ln\left(\frac{H_1}{H_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

But soft! What light through yonder window breaks? After a little rearrangement, we find:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

Remember, δ here is just a number that we found by taking the natural log of two amplitudes, it is a physical value that we already know!

The experiment in part (b) of Figure [3], published by Chen et al, used real-life data to find a damping ratio of $\zeta = 0.215 \pm 0.013$. This is quite nice because we assumed that our system was underdamped ($0 < \zeta < 1$), which turned out to be true.

Is this not simply magnificent? Using just the ratio of consecutive amplitudes of decaying oscillation, we can find our damping ratio ζ. Indeed, we have arrived at a beautiful result.

5 Kelvin-Voigt: supermodel

It is quite satisfying to know the damping ratio, but there must be some greater application that we would want it for. After all, why else would we do all this work just to find some number?

To explore the myriad uses of the damping ratio, we'll look to the world of continuum mechanics. Long gone are the days of wonderfully linear stress-strain relations and Hooke's law. In this section, we'll splash into the shallows of the vast sea of viscoelastic material characterization.

On a stress-strain curve, we'll first consider the linear elastic region, which experiences elastic behavior (see the left graph in Figure [4]). When an applied strain ϵ is removed, the stress σ in the structure goes back to zero, and no energy is dissipated in the process of load application. This is the result of bond stretching along crystallographic planes, and luckily for us, unless the material reaches its yield strength, elastic materials go right back to where they started (otherwise, things fall apart). These Newtonian materials, or materials that have a linear relationship between stress and strain, observe no time dependence. This does not hold for all materials, which is why we have to go through this little detour to get back to the damping ratio.

Human tissue, for example, does not behave with linear elasticity like metals do in certain stress ranges. Instead, human tissues behave in a manner consistent with viscoelastic materials. Viscoelastic materials are time dependent, meaning that the rate at which a strain is applied will change the slope of the loading curve. A higher strain rate may lead to a steeper stress-strain curve. Viscoelastics also dissipate energy when a load is applied and then removed, resulting in *hysteresis* in the stress-strain curve. When loaded, the viscoelastic material (the right graph in Figure [4]) stores energy, and upon unloading, energy is recovered. The area between the two curves represents the energy lost through heat and deformation mechanisms, which makes viscoelastic materials excellent for vibration absorption and insulation - and for our bodies.

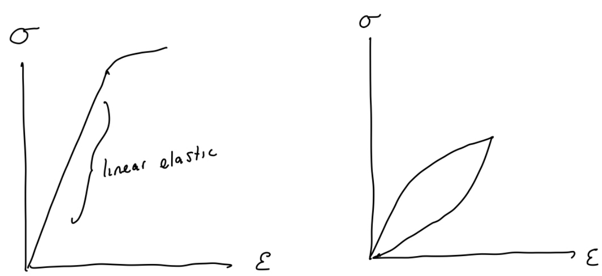


Figure 4: Elastic vs Viscoelastic Behavior

We aim to model viscoelastic materials with rheological models that combine an elastic component and a viscous component. Typically, a linear elastic (Hookean) spring is used to represent the elastic contribution, and a linear viscous damper (Newtonian dashpot) is used to represent the effects of damping through viscous friction.

The *Hookean spring's* mechanical contribution can be expressed with a version of Hooke's law:

$$\sigma = E\epsilon \tag{19}$$

Where E is the Elastic Modulus of the material (similar to the spring constant k), ϵ is the unitless strain, and σ is the stress.

The *Newtonian dashpot*, which produces a stress proportional to the strain-rate, can be quantified as:

$$\sigma = \eta\dot{\epsilon} \tag{20}$$

Where η is the viscosity of the material and $\dot{\epsilon}$ is the rate at which strain is applied, analogous to the damping coefficient c and velocity $\dot{x}$, respectively.

For our purposes in modeling a SDOFHO, we'll use the dimensionless damping ratio ζ, or the ratio of the viscous parameter to the critical damping parameter, in place of viscosity η.

These two building blocks are usually combined in one of two ways: the Maxwell model and the Kelvin-Voigt model. The Maxwell model assumes that the spring and dashpot are connected in *series* and when the ends are pulled, they experience the same stress, and the total strain is equal to the sum of their individual strains.

We can quantify the Maxwell Model by differentiating and rearranging the total strain equation:

$$\frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = \dot{\epsilon} \tag{21}$$

This configuration, unfortunately, does not quite model what we're looking for in our SDOFHO. Having already connected our spring and dashpot in series, we'll look to a different connection schema: connecting them in *parallel.* This is called the Kelvin-Voigt (or just Voigt) model, and upon inspection, we find that it is *exactly* what we are looking for!

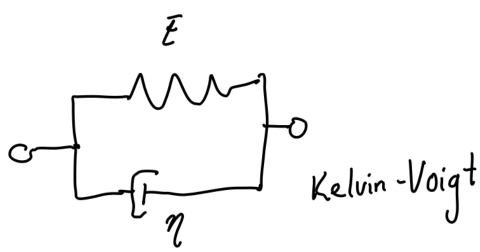


Figure 5: The Voigt Model looks suspiciously like the SDOFHO we drew earlier

The Voigt model, consisting of a spring and dashpot connected in parallel, indicates that both components feel the same strain, and the total stress is the sum of the stresses of the individual components.

$$\sigma(t) = E\epsilon(t) + \eta\dot{\epsilon}(t) \tag{22}$$

Our Voigt model will use the damping ratio ζ in place of the viscosity η above. We now posses a constitutive model of linear viscoelasticity that has the potential to characterize human tissue *and* incorporate the damping ratio. What can we do with it?

6 Breast practices for free vibration

Putting theory into practice, let's examine a breast. The biomechanical modelling of the human breast is no trivial task, but it remains an important one to billions of people around the world.

In their 2013 paper in *Ergonomics*, 'A study of breast motion using nonlinear dynamic FE (finite element) analysis', Chen et al. note that "excessive breast motion can induce breast pain in women and exercise-associated breast pain is positively related to breast displacement.[1]" No duh.

Dr. Deirdre McGhee, from the University of Wollongong, found that large breasts can generate enough force to break a clavicle[2], and Dr. Alex Milligan, from the Univeristy of Portsmouth, found that breast pain affects 72% of exercising females[3], likely caused by tension on the skin, fascia, and nerves during periods of high breast displacement.

Breast modelling with FE is still an emerging field due to the inhomogeneity of breast tissue, potential pain and embarrassment of volunteers, and the lack of concrete knowledge of boundary conditions and material parameters for FE models. Developing more accurate models of viscoelastic deformation is key to using software to help analyze and design for today's breasts, working toward eliminating unnecessary pain.

Because they consist of varying layers of fat, glandular tissue, skin, fascia, and muscle, breasts require immense and virtually nonexistent levels of computing power for accurate finite element simulations. Despite insufficient computational resources, we can attack complex problems by making simplifying assumptions, developing a basic model, and proceeding with basic analyses. Only after all that can we even start to think about dumping our problem in a computer and letting it chug away.

As an old saying goes: *"There's only one way to eat an elephant: one bite at a time."*

Of the many, many interacting branches of science that go into biomechanical modelling, we can focus on what we know: a single vibrating mass. We can use Newton's Second Law, ordinary differential equations, and the logarithmic decrement to calculate the damping ratio of any SDOFHO!

How then, do we measure the damping ratio ζ of a human breast? It's simple: let it go. A vertical breast drop is an example of damped vibration, and the oscillations of the breast can be mapped in the time domain to gather a free-vibration displacement history.

A simple way to do this, as used by Chen et al. and many others, uses a camera and a retro-reflective marker. The marker is placed at the point of maximum deflection (typically the nipple), and the subject raises and drops their

[1] Chen, L., Ng, S., Yu, W., Zhou, J., Wan, K. (2013). A study of breast motion using nonlinear dynamic FE analysis. Ergonomics, 56(5), 868-878. doi: 10.1080/00140139.2013.777798

[2] McGhee, D. (2009). Sports bra design and bra fit: minimising exercise-induced breast discomfort (Ph.D). University of Wollongong.

[3] Milligan, A. (2013). The effect of breast support on running biomechanics (Ph.D). University of Portsmouth.

breast, resulting in free decay, or amplitude attenuation by viscous damping. Plotting the vertical position (using a measuring stick) versus time, the result is a graph[4] that looks like the one below.

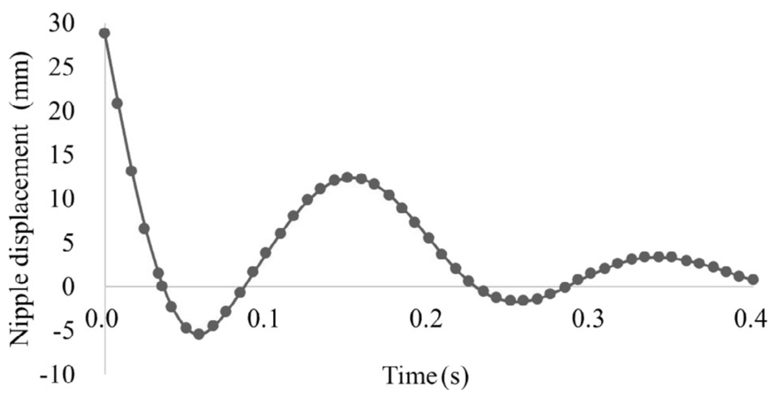


Figure 6: Nipple displacement plotted against time for a freely vibrating breast

Interestingly enough, this plot looks familiar. Using the method of the logarithmic decrement, we can find the damping ratio of the breast ζ_{breast} using only the amplitudes of the first two peaks and some clever mathematics.

The value of $\zeta = 0.215$ from Chen et al. is typical of a 36B bra size, but much variation is expected. Depending on the size of the breasts, the age of the breasts, the person to whom the breasts belong, and a variety of other factors, we can expect a damping ratio somewhere in the 0.2 to 0.5 range. These damping ratios can be used to help increase the complexity of FE models for viscoelastic deformation and further our understanding of biomechanical modelling in the context of breast pain. While this number may not be the exact damping ratio of the incredibly complex viscoelastic continuum that makes up the breast, it provides an approximation that is easy to test, to measure, and to calculate. What more could you ask for from an approximation? Besides, for most people, approximations are already more than good enough.

[4]Cai, Y., Chen, L., Yu, W., Zhou, J., Wan, F., Suh, M., Chow, D. (2018). A piecewise mass-spring-damper model of the human breast. Journal Of Biomechanics, 67, 137-143. doi: 10.1016/j.jbiomech.2017.11.027

7 Conclusion

From a basic harmonic oscillator to the millions of nodes on a finite element breast, mechanics pervades every aspect of physical simulation and modelling. Principles of mechanics can be seen at work in the construction of our bodies, heard in the thrumming of jet engines, and felt in the cool of our air-conditioned houses. Though it doesn't always make sense at first, mechanics can shed valuable insight on problems that seem insurmountable.

The mechanics of vibration is not child's play, and most analytical solutions, even ones for systems with only 1 degree of freedom, can be tricky, relying on a strong grasp of ordinary differential equations and physics. Similarly, continuum mechanics and rheology stand atop a wall of tensor algebra and calculus, and viscoelasticity incorporates tools from statistical thermodynamics and polymer science.

Unsurprisingly, the world of mechanics does not stop there, and it can tell us more about ourselves and our environments than we could possibly think to ask. From vibrating masses to breast damping, never stop questioning and analyzing the mathematical world around you. The extent of your curiosity predicates the extent of your wonder, and what is life without wonderful things?

-Nicholas Ong

FIN.